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Multiplicity A_m models*

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Abstract. Models generalizing the $su(2)$ XX spin-chain were recently introduced. These XXC models also have an underlying $su(2)$ structure. Their construction method is shown to generalize to the chains based on the fundamental representations of the A_m Lie algebras. Integrability of the new models is shown in the context of the quantum inverse scattering method. Their R-matrix is found and shown to yield a representation of the Hecke algebra. The diagonalization of the transfer matrices is carried out using the algebraic Bethe Ansatz. I comment on eventual generalizations and possible links to reaction-diffusion processes.

PACS. 02.90.+p Other topics in mathematical methods in physics – 05.50.+q Lattice theory and statistics; Ising problems – 75.10.Jm Quantized spin models

1 Introduction

In the course of generalizing the Hubbard model new models were discovered: the so-called XXC models [1–4]. These one-dimensional (spin-chain) integrable models have a natural expression in terms of $su(n)$ generators, rather than higher-spin representations of $su(2)$. The first Hamiltonian of the integrable hierarchy is bilinear in terms of $su(n)$ generators. The symmetries of the models are generators of $su(n_1) \oplus su(n_2) \oplus u(1)$. The algebraic Bethe Ansatz diagonalization of the transfer matrices requires a nesting similar to the XXZ models based on the fundamental representation of $su(p)$. Finally, for a subclass of the XXC models the explicit form of all the conserved charges was found [1]; their expressions in terms of $su(n)$ structure constants indicate an $su(n)$ interpretation. New identities for the structure constants were derived as a by-product. These should admit further generalizations.

However the XXC models also share features associated with the spin-1/2 XXZ model and appear as a kind of higher-dimensional representations of the R-matrix of the spin-1/2 model. Indeed, the latter model is a special case of the XXC models and the R-matrices of these models share a common structure, with their building blocks satisfying the same algebraic relations. Moreover the algebraic Bethe Ansatz can also be interpreted as being nestless and therefore a simple generalization of the $su(2)$ one. Another argument in favor of the $su(2)$ interpretation can be found in [5]. One obtains the quadratic Hamiltonian of a particular XXC model, with open boundary conditions, as the "infinite-coupling" restriction of the Hubbard Hamiltonian on a subspace of the complete Hilbert space. The authors of [5] showed that this model possessed a surprisingly large affine symmetry based on $su(2)$. A generalization of the resulting model yielded a subclass of open-boundaries XXC Hamiltonians; however their symmetries favor an $su(n)$ interpretation.

One therefore may try to generalize the construction method used in deriving the XXC models, to the XXZ models which are based on the fundamental representations of the A_m Lie algebras. This approach turns out to work and the resulting models are obtained and studied in this paper. One starts with the A_m R-matrices. Their structure allows a straightforward generalization retaining their operatorial form and their A_m characteristics. The new matrices are given and shown to satisfy the Yang-Baxter equation. Integrability of the models is then a simple consequence of the quantum inverse scattering framework. The symmetries are obtained and the transfer matrices are diagonalized by algebraic Bethe Ansatz. I conclude with some remarks and possible physical applications to reaction-diffusion processes.

2 New models

The Yang-Baxter equation (YBE) is at the center of Quantum Inverse Scattering Method (QISM) used to obtain quantum integrable one-dimensional spin-chains and their covering two-dimensional classical statistical models [6–9]. There are now several methods which can be used to obtain solutions, R- or L-matrices, of the YBE. An important and quite general method relies on the use of affine quantum groups based on Lie algebras. Rather than directly solve the cubic equations resulting from the YBE,

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one solves linear equations where R appears as the intertwiner between two possible deformed coproducts (tensor products). This method was used in particular in [10] and explicit trigonometric solutions were obtained for the fundamental representations of the classical Lie algebras A_m, B_m, C_m, D_m, and their twisted versions.

For the untwisted algebra A_{m-1}, the trigonometric $\check{R}$-matrix of the fundamental representation is m^2-dimensional and can be found in [10]:

$$\check{R}(y) = \sin(\gamma)(y\sum_{\alpha>\beta} E^{\beta\beta}\otimes E^{\alpha\alpha} + y^{-1}\sum_{\alpha<\beta} E^{\beta\beta}\otimes E^{\alpha\alpha}) + \sin(\lambda+\gamma)\sum_{\alpha} E^{\alpha\alpha}\otimes E^{\alpha\alpha} + \sin(\lambda)\sum_{\alpha\neq\beta} E^{\beta\alpha}\otimes E^{\alpha\beta} \quad (1)$$

where $y = e^{i\lambda}$, λ is the spectral parameter and γ the quantum deformation parameter. The $E^{\alpha\beta}$ are $m\times m$ matrices with a one at row α and column β and zeros otherwise. The matrix $\check{R}$ satisfies the Yang-Baxter equation

$$\check{R}_{12}(\lambda)\check{R}_{23}(\lambda+\mu)\check{R}_{12}(\mu) = \check{R}_{23}(\mu)\check{R}_{12}(\lambda+\mu)\check{R}_{23}(\lambda) \quad (2)$$

for any fixed value of γ. Here and in (12), the notation $\mathcal{O}_{ij}$ $(i\neq j)$ means that the operator $\mathcal{O}$ acts non-trivially on the ith and jth spaces, and as the identity on the other spaces. The regularity and unitarity properties also hold:

$$\check{R}(0) = \mathbb{I}\sin\gamma,$$
$$\check{R}(\lambda)\check{R}(-\lambda) = \mathbb{I}\,\sin(\gamma+\lambda)\sin(\gamma-\lambda). \quad (3)$$

I now give new solutions to the YBE which are obtained by a multi-state generalization of expression (1). One first rewrites the latter matrix as

$$\check{R}(\lambda) = (yP^{(+)} + y^{-1}P^{(-)})\sin\gamma + P^{(2)}\sin(\lambda+\gamma) + P^{(3)}\sin\lambda \quad (4)$$

and looks for representations of the operators P which allow $\check{R}$ to satisfy the Yang-Baxter equation. Let n_i be m positive integers such that

$$\sum_{i=1}^{m} n_i = n \text{ and } 1\leq n_1\leq ...\leq n_m\leq n-1. \quad (5)$$

The inequality restrictions avoid multiple counting of models. Split the set of n basis states into m disjoint sets $\mathcal{A}_i$:

$$\text{card}\,(\mathcal{A}_i) = n_i,\ \mathcal{A}_i\cap\mathcal{A}_j = \emptyset \quad \text{for } i\neq j \quad (6)$$

$\mathcal{A}_i$ should not be confused with the Lie algebra $su(i+1)$. Consider the following expression for $P^{(3)}$:

$$P^{(3)} = \sum_{1\leq i<j\leq m}\sum_{\alpha_i\in\mathcal{A}_i}\sum_{\alpha_j\in\mathcal{A}_j}\Big(x_{\alpha_i\alpha_j}E^{\alpha_i\alpha_j}\otimes E^{\alpha_j\alpha_i} + x^{-1}_{\alpha_i\alpha_j}E^{\alpha_j\alpha_i}\otimes E^{\alpha_i\alpha_j}\Big). \quad (7)$$

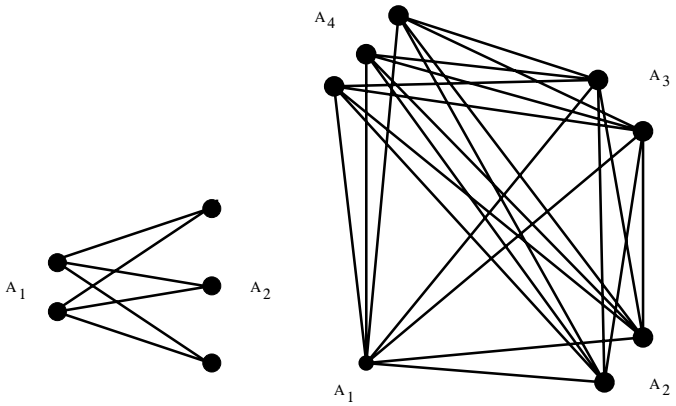

Fig. 1. The diagram on the left corresponds to the system $(2,3;2,5)$ and the one on the right to $(1,2,2,3;4,8)$. Here A_i stands for $\mathcal{A}_i$.

The twist parameters $x_{\alpha_i\alpha_j}$ are arbitrary complex numbers. The remaining operators are given by:

$$P^{(1)} \equiv (P^{(3)})^2 = P^{(+)} + P^{(-)} = \sum_{1\leq i<j\leq m}\sum_{\alpha_i\in\mathcal{A}_i}\sum_{\alpha_j\in\mathcal{A}_j}(E^{\alpha_i\alpha_i}\otimes E^{\alpha_j\alpha_j} + E^{\alpha_j\alpha_j}\otimes E^{\alpha_i\alpha_i}) \quad (8)$$

$$P^{(2)} \equiv \mathbb{I} - P^{(1)} = \sum_{i=1}^{m}\sum_{\alpha_i\in\mathcal{A}_i}\sum_{\beta_i\in\mathcal{A}_i} E^{\alpha_i\alpha_i}\otimes E^{\beta_i\beta_i}. \quad (9)$$

The operators $P^{(+)}$ and $P^{(-)}$ correspond respectively to the sums over the first and second summands in (8). There is also another way of writing (4):

$$\check{R}(\lambda) = \mathbb{I}\sin(\lambda+\gamma) + P\sin\lambda \quad (10)$$
$$P \equiv P^{(3)} - (e^{-i\gamma}P^{(+)} + e^{i\gamma}P^{(-)}). \quad (11)$$

A straightforward if tedious calculation shows that:

$$P^2 = -2P\cos\gamma,$$
$$P_{12}P_{23}P_{12} + P_{23} = P_{23}P_{12}P_{23} + P_{12}. \quad (12)$$

P is therefore a generator of the Hecke algebra. These relations imply that the Yang-Baxter equation is satisfied. The regularity and unitarity properties (3) still hold. I denote this model by $(n_1,...,n_m;m,n)$.

There is a simple graphic and mnemonic representation of the foregoing operators. To each state assign a point in the plane. States belonging to the same set $\mathcal{A}_i$ are not linked while those belonging to different sets are linked. A given link corresponds to a given summand appearing in the expression of $P^{(3)}$, a tensor product of two step operators, and also to the summands appearing in $P^{(\pm)}$, a tensor product of diagonal operators. Similarly the summand of $P^{(2)}$ corresponds to missing links in the diagram. Links and missing links exhaust all possible links which could be drawn between the n states. This representation is illustrated with two examples in Figure 1. One can also read this diagram as follows. One starts with an

$su(m)$ system and replaces every state with an arbitrary number of copies. The copies originating from the same state do not "interact" among each other; they interact with all other states and their copies as dictated by the original diagram.

The choice of which states go into which set A_i does not yield inequivalent systems. For instance the two systems, $(\mathcal{A}_1 = \{1\},\ \mathcal{A}_2 = \{2,3\})$ and $(\mathcal{A}_1 = \{2\},\ \mathcal{A}_2 = \{1,3\})$, are related by a simple (orthogonal) permutation matrix whose N-fold tensor product with itself yields the unitary matrix which relates the two N-sites integrable models [1].

The operators $P^{(1)}$ and $P^{(2)}$ form a complete set of projectors on the tensor product space $\mathbb{C}^n \otimes \mathbb{C}^n$:

$$\begin{aligned} P^{(1)} + P^{(2)} &= \mathbb{I}, \\ (P^{(1)})^2 &= P^{(1)}, \\ (P^{(2)})^2 &= P^{(2)}, \\ P^{(1)}P^{(2)} &= P^{(2)}P^{(1)} = 0. \end{aligned} \tag{13}$$

One also has $(P^{(3)})^3 = P^{(3)}$. However, for $m > 2$, the operator $P^{(3)}$ does not satisfy the 3-sites relations of the "free-fermions" algebra $\mathcal{A}$ found in [3]. This is an important difference between $m = 2$ and $m > 2$. In the latter case no "conjugation matrix" exists and it does not seem possible to couple two such models.

For $m = 2$ one obtains the XXC models in their asymmetric guise, as there are factors of y and y^{-1}. The transformation to the symmetric models of [4] is given by a simple "gauge" transformation:

$$\check{R}^{GT}(\lambda) = (A(\lambda) \otimes \mathbb{I})\check{R}(\lambda)(\mathbb{I} \otimes A(-\lambda)) \tag{14}$$

where $A(\lambda) = \sum_{\alpha_1} E^{\alpha_1\alpha_1} e^{i\lambda c_1} + \sum_{\alpha_2} E^{\alpha_2\alpha_2} e^{i\lambda c_2}$ with $c_2 - c_1 = 1$. All the above properties are preserved by such a transformation. For $m > 2$ however it is not possible to remove the $y^{\pm 1}$ factors. For $m = n$ and all parameters x equal to one, one obtains the matrix (1).

The rational limit of $\check{R}$ is obtained by letting $\lambda \to \gamma\lambda$, dividing by $\sin\gamma$ and taking the limit $\gamma \to 0$. These manipulations conserve all the properties of the $\check{R}$-matrix. In particular, one obtains

$$\check{R}(\lambda) = P^{(1)} + (1+\lambda)P^{(2)} + \lambda P^{(3)} \tag{15}$$

$$\check{R}(0) = \mathbb{I}, \quad \check{R}(\lambda)\check{R}(-\lambda) = \mathbb{I}(1-\lambda^2). \tag{16}$$

If all parameters $x_{\alpha_i\alpha_j}$ are equal to each other, both the trigonometric and rational matrix have the following symmetry

$$\begin{aligned} &[M \otimes M, \check{R}(\lambda)] = 0, \\ &M = \sum_{k=1}^{m} M^{(k)} = \sum_{k=1}^{m} \sum_{\alpha_k,\beta_k \in \mathcal{A}_k} m^{(k)}_{\alpha_k\beta_k} E^{\alpha_k\beta_k}. \end{aligned} \tag{17}$$

If however two blocks $M^{(k)}$ and $M^{(k')}$, $k < k'$, are diagonal M is a symmetry of R with the corresponding parameters $x_{\alpha_k\alpha_{k'}}$ being unconstrained.

The transfer matrix is the generating functional of the infinite set of conserved quantities. Its construction in the framework of the Quantum Inverse Scattering Method is well-known. The Lax operator on a chain at site i with inhomogeneity μ_i is given by:

$$L_{0i}(\lambda) = R_{0i}(\lambda - \mu_i) = \mathcal{P}_{0i}\check{R}_{0i}(\lambda - \mu_i) \tag{18}$$

where $\mathcal{P}$ is the permutation operator on $\mathbb{C}^n \otimes \mathbb{C}^n$. The monodromy matrix is a product of Lax operators

$$T(\lambda) = M_0 L_{0N}(\lambda)...L_{01}(\lambda) \tag{19}$$

where N is the number of sites on the chain and 0 is the auxiliary space. The transfer matrix is the trace of the monodromy matrix over the auxiliary space: $\tau(\lambda) = \mathrm{Tr}_0\,[T(\lambda)]$. The introduction of M corresponds to integrable periodic M-twisted boundary conditions. A set of local conserved quantities is given by

$$H_{p+1} = \left(\frac{d^p \ln\tau(\lambda)}{d\lambda^p}\right)_{\lambda=0}, \quad p \geq 0. \tag{20}$$

The YBE implies the following intertwining relations for the elements of the monodromy matrix:

$$\begin{aligned} &\check{R}(\lambda_1 - \lambda_2)T(\lambda_1) \otimes T(\lambda_2) = \\ &\qquad T(\lambda_2) \otimes T(\lambda_1)\check{R}(\lambda_1 - \lambda_2). \end{aligned} \tag{21}$$

Taking the trace over the auxiliary spaces, and using the cyclicity property of the trace, one obtains $[\tau(\lambda_1), \tau(\lambda_2)] = 0$. The Hamiltonians H_p therefore mutually commute.

The quadratic Hamiltonian calculated from (20), for $\mu_i = 0$ and $M = \mathbb{I}$, is equal to

$$\begin{aligned} H_2 = \sum_j H_{jj+1} = \frac{1}{\sin\gamma}\sum_j \Big(&P^{(3)}_{jj+1} + P^{(2)}_{jj+1}\cos\gamma \\ &+ (P^{(+)}_{jj+1} - P^{(-)}_{jj+1})i\sin\gamma\Big). \end{aligned} \tag{22}$$

For $|x_{a\beta}| = 1$ and γ purely imaginary the Hamiltonian $H_2 \sin\gamma$ is hermitian. For $m = 2$ this Hamiltonian is also hermitian for real values of γ. Under the periodic boundary conditions the non-hermitian part does not contribute. This is easily seen from the transformation (14) and the fact that the Hamiltonian density H_{jj+1} is equal to the derivative at zero of $\check{R}(\lambda)$. The rational limit yields: $H_2 = \sum_j (P^{(3)}_{jj+1} + P^{(2)}_{jj+1})$, and, provided $|x_{a\beta}| = 1$, the Hamiltonians are hermitian.

The cubic conserved quantity is obtained from (20) by a direct calculation. One finds $H_3 = -\sum_j [H_{j-1j}, H_{jj+1}] - (N/\sin^2\gamma)\mathbb{I}$, where H_{jj+1} is the Hamiltonian density of (22) or its rational version. The $\sin^2\gamma$ is replaced by 1 for the rational limit. The commutator can be easily derived using $E^{\alpha\beta}E^{\gamma\delta} = \delta_{\beta\gamma}E^{\alpha\delta}$. No general closed form expressions for the higher conserved quantities have yet been derived in the literature; however see [1] for some specific cases, and the references therein for related issues.

$$T_{\alpha_{k_0^{(1)}}\alpha_{k_0^{(1)}}} C_{\delta_{k_1}^{(1)}}(\lambda_1)...C_{\delta_{k_{p_1}}^{(p_1)}}(\lambda_{p_1})\,||\gamma_{k_0^{(1)}}\rangle = \delta_{p_1 N}\prod_{i=1}^{N} x^{\mathrm{sign}(k_0^{(1)}-k_i)}\frac{\sin(\lambda-\mu_i)}{\sin(\lambda-\mu_i+\gamma)}\; C_{\delta_{k_1}^{(1)}}(\lambda_1)...C_{\delta_{k_{p_1}}^{(p_1)}}(\lambda_{p_1})\,||\gamma_{k_0^{(1)}}\rangle \qquad (31)$$

The transfer matrix, and therefore all the conserved quantities, have the symmetries of the $\check{R}$-matrix. Define the magnetic-field operators as:

$$H_1^{\alpha_k\beta_k} \equiv \sum_i E_i^{\alpha_k\beta_k}\alpha_k, \beta_k \in \mathcal{A}_k. \qquad (23)$$

One has the following commutation relations for both the trigonometric and rational forms:

$$\begin{aligned}
&[H_1^{\alpha_k\beta_k}, \tau(\lambda)] = 0 \text{ if and only if}\\
&\forall j<k,\ \forall\gamma_j\in\mathcal{A}_j\ \ x_{\gamma_j\alpha_k} = x_{\gamma_j\beta_k}\\
&\text{and } \forall j>k,\ \forall\gamma_j\in\mathcal{A}_j\ x_{\alpha_k\gamma_j} = x_{\beta_k\gamma_j}\\
&\text{and } m^{(k)}_{\alpha_k\alpha_k} = m^{(k)}_{\beta_k\beta_k},\ \ m^{(k)}_{\gamma_k\alpha_k} = 0\ \ \forall\gamma_k\neq\alpha_k,\\
&m^{(k)}_{\beta_k\gamma_k} = 0\ \ \forall\gamma_k\neq\beta_k. \qquad (24)
\end{aligned}$$

In particular the diagonal operators $H_1^{\alpha_k\alpha_k}$ commute with the transfer matrix without any constraint on the twist parameters, but with the above constraints on the matrix M. The *rational* transfer matrix, with all $x_{\alpha_i\alpha_j}$'s equal to one, may have additional symmetries. For $k\neq k'$ one finds

$$\begin{aligned}
&[H_1^{\alpha_k\beta_{k'}}, \tau^{rat}(\lambda)] = [H_1^{\beta_{k'}\alpha_k}, \tau^{rat}(\lambda)] = 0\\
&\text{if and only if } n_k = n_{k'} = 1 \text{ and } m^{(k)}_{\alpha_k\alpha_k} = m^{(k')}_{\beta_{k'}\beta_{k'}}. \qquad (25)
\end{aligned}$$

In the following we shall concentrate on the case where $M=\mathbb{I}$ and all the x's are equal to a single parameter x, despite the fact some results hold for generic parameters. The full local symmetry then is $su(n_1)\oplus...\oplus su(n_m)\oplus u(1)\oplus...\oplus u(1)$ where there are $m-1$ $u(1)$'s.

3 Algebraic Bethe Ansatz

The diagonalization by algebraic Bethe Ansatz of the foregoing models combines features from the diagonalization of the $(1,...,1;m,m)$ models and the XXC models. We refer the reader to [4,8,11] for some details and give here the new features and results. The $\check{R}$-matrix here is redefined as the matrix (4) divided by $\sin(\lambda+\gamma)$. This adds $-N\cot\gamma\,\mathbb{I}$ to the Hamiltonian (22) and $+(N/\cos^2\gamma)\mathbb{I}$ to H_3. For the rational limit one adds $-N\mathbb{I}$ and $+N\mathbb{I}$ respectively.

Let $k_0^{(1)}\in\{1,...,m\}$ and $\gamma_{k_0^{(1)}}\in\mathcal{A}_{k_0^{(1)}}$ be given. The action of all the elements of the monodromy matrix on the pseudo-vacuum $||\gamma_{k_0^{(1)}}\rangle \equiv |\gamma_{k_0^{(1)}}\rangle\otimes...\otimes|\gamma_{k_0^{(1)}}\rangle$ is easily derived:

$$\begin{aligned}
&T_{\gamma_{k_0^{(1)}}\alpha_k}\,||\gamma_{k_0^{(1)}}\rangle \neq 0,\\
&T_{\gamma_{k_0^{(1)}}\alpha_{k_0^{(1)}}}\,||\gamma_{k_0^{(1)}}\rangle \neq 0 \qquad (26)\\
&T_{\gamma_{k_0^{(1)}}\gamma_{k_0^{(1)}}}\,||\gamma_{k_0^{(1)}}\rangle = ||\gamma_{k_0^{(1)}}\rangle,\\
&T_{\gamma_k\gamma_k}||\gamma_{k_0^{(1)}}\rangle = \prod_{i=1}^{N}\left(x^{\mathrm{sign}(k-k_0^{(1)})}\frac{\sin(\lambda-\mu_i)}{\sin(\lambda-\mu_i+\gamma)}\right)||\gamma_{k_0^{(1)}}\rangle \qquad (27)
\end{aligned}$$

$\forall k\neq k_0^{(1)}$ and $\forall\alpha_{k_0^{(1)}}\neq\gamma_{k_0^{(1)}}$. All other elements of T annihilate this vector.

Let $C_{\beta_k}\equiv T_{\gamma_{k_0^{(1)}}\beta_k}$. Operator C_{β_k} exactly flips a state $|\gamma_{k_0^{(1)}}\rangle$ into a state $|\beta_k\rangle$. Thus these operators acting on $||\gamma_{k_0^{(1)}}\rangle$ give a linear combination of states where exactly one state in $||\gamma_{k_0^{(1)}}\rangle$ has been changed to $|\beta_k\rangle$, at every site. To show this one uses the following relations:

$$[H_1^{\alpha_k\alpha_k}, C_{\beta_{k'}}] = \delta_{\alpha_k\beta_{k'}}C_{\beta_{k'}},\ \forall k,\ \forall k'\neq k_0^{(1)} \qquad (28)$$

$$[H_1^{\alpha_{k_0^{(1)}}\alpha_{k_0^{(1)}}}, C_{\beta_k}] = 0,\ \forall k\neq k_0^{(1)}, \forall\alpha_{k_0^{(1)}}\neq\gamma_{k_0^{(1)}} \qquad (29)$$

$$[H_1^{\gamma_{k_0^{(1)}}\gamma_{k_0^{(1)}}}, C_{\beta_k}] = -C_{\beta_k},\ \forall k\neq k_0^{(1)}. \qquad (30)$$

Relation (30) also shows that $C_{\delta_{k_1}^{(1)}}(\lambda_1)...C_{\delta_{k_{p_1}}^{(p_1)}}(\lambda_{p_1})\,||\gamma_{k_0^{(1)}}\rangle = 0$ for $p_1 > N$. Relation (29) shows that this same vector has no $|\alpha_{k_0^{(1)}}\rangle$ state in it, and (29, 30) yield for $\alpha_{k_0^{(1)}}\neq\gamma_{k_0^{(1)}}$:

see equation (31) above.

The operators C_{δ_k} are therefore candidates for writing the eigenvector Ansatz. In contrast, the operator $T_{\gamma_{k_0^{(1)}}\alpha_{k_0^{(1)}}}$ acting on $||\gamma_{k_0^{(1)}}\rangle$ changes the state of only the *first* site to $|\alpha_{k_0^{(1)}}\rangle$. This is an unusual feature and these operators cannot be used to write down an eigenvector Ansatz.

One may therefore take as Bethe Ansatz eigenvector

$$|\lambda_1,...,\lambda_{p_1}\rangle \equiv F^{\delta_{k_1}^{(1)},...,\delta_{k_1}^{(p_1)}} C_{\delta_{k_1}^{(1)}}(\lambda_1)...C_{\delta_{k_{p_1}}^{(p_1)}}(\lambda_{p_1})\,||\gamma_{k_0^{(1)}}\rangle \qquad (32)$$

where the parameters λ_i and the coefficients F are to be determined. The sums run over all k_i from 1 to m with $k_i\neq k_0^{(1)}$ and over $\delta_{k_i}^{(i)}$ in $\mathcal{A}_{k_i}$.

$$\Lambda^{(m-k;p_k)}(\lambda,\{\lambda_1^{(k)},...,\lambda_{p_k}^{(k)}\}) = \delta_{p_{k+1}p_k}\,(n_{q_k}-1)\prod_{i=1}^{p_k}\left(\frac{x^\epsilon\sin(\lambda-\lambda_i^{(k)})}{\sin(\lambda-\lambda_i^{(k)}+\gamma)}\right)+\prod_{i=1}^{p_{k+1}}\left(\frac{x^\epsilon\sin(\lambda_i^{(k+1)}-\lambda+\gamma)}{\sin(\lambda_i^{(k+1)}-\lambda)}\right)$$
$$+\prod_{i=1}^{p_k}\left(\frac{x^{-\epsilon}\sin(\lambda-\lambda_i^{(k)})}{\sin(\lambda-\lambda_i^{(k)}+\gamma)}\right)\prod_{j=1}^{p_{k+1}}\left(\frac{x^\epsilon\sin(\lambda-\lambda_j^{(k+1)}+\gamma)}{\sin(\lambda-\lambda_j^{(k+1)})}\right)\Lambda^{(m-k-1;p_{k+1})}(\lambda,\{\lambda_1^{(k+1)},...,\lambda_{p_{k+1}}^{(k+1)}\}) \quad (36)$$

$(\lambda_1^{(0)},...,\lambda_N^{(0)}) = (\mu_1,...,\mu_N), \quad k=0,...,m-2$

One then applies the transfer matrix on the state $|\lambda_1,...,\lambda_p\rangle$ and uses the algebraic relations (21). The foregoing procedure, the nested algebraic Bethe Ansatz, is a cumbersome but straightforward generalization of the one for the usual $(1,...,1;m,m)$ models, *i.e.* the $su(m)$ XXZ model. The differences come from the sum on the multiple states in each set $\mathcal{A}_i$, as already seen on the initial eigenvector Ansatz (32). The are $m-1$ levels in the nesting and diagonalizing the transfer matrix at one level requires diagonalizing a new transfer matrix generated by the repeated use of relations (21). This new transfer matrix corresponds to a system of the above type but with a reduced number of states and sites. The nesting stops at the last level at which the new transfer matrix is trivially diagonal. Technical considerations impose a decreasing or increasing sequence of $k_0^{(1)}$, one being needed for every level of the Ansatz. For the increasing sequence $(k_0^{(1)}=1, k_0^{(2)}=2,...,m-1)$, $\epsilon=-1$ in the eigenvalue and Bethe Ansatz equations while for the decreasing sequence $\epsilon=+1$. The sequence of systems appearing for the increasing and decreasing sequences are given by

$$\begin{aligned}&(n_1,...,n_m;m,n) \text{ for } p_0=N \text{ sites} \rightarrow\\ &(n_2,...,n_m;m-1,n-n_1) \text{ for } p_1 \text{ sites} \rightarrow\\ &\cdots\rightarrow(n_{m-1},n_m;2,n-n_1-\cdots-n_{m-2}) \text{ for } p_{m-2} \text{ sites}\end{aligned} \quad (33)$$

$$\begin{aligned}&(n_1,...,n_m;m,n) \text{ for } p_0=N \text{ sites} \rightarrow\\ &(n_1,...,n_{m-1};m-1,n-n_m) \text{ for } p_1 \text{ sites} \rightarrow\\ &\cdots\rightarrow(n_1,n_2;2,n-n_3-\cdots-n_m) \text{ for } p_{m-2} \text{ sites.}\end{aligned} \quad (34)$$

The sequence of sites is non-increasing:

$$N=p_0\geq p_1\geq\cdots\geq p_{m-1}\geq 0. \quad (35)$$

The p_{m-1} appears in the diagonalization of the last system, $(*,*;2,*)$ in the above series; this system is an XXC one. The above procedure is akin to decimating the A_{m-1} Dynkin diagram by going from either of its extremities to the other.

The eigenvalue at one level is related to the eigenvalue at the following level as follows:

see equation (36) above

where the subscript q_k appearing in the δ-term is equal to $k+1$ $(m-k)$ for $\epsilon=-1$ $(\epsilon=+1)$ respectively. The eigenvalue of the transfer matrix $\tau(\lambda)$ is $\Lambda^{(m;p_0)}(\lambda,\{\lambda_1^{(0)},...,\lambda_N^{(0)}\})$. The last eigenvalue $\Lambda^{(1;p_{m-1})}(\lambda,\{\lambda_1^{(m-1)},...,\lambda_{p_{m-1}}^{(m-1)}\})$ is independent of the spectral parameter and of the inhomogeneities. It is an eigenvalue of the (constant) unit-shift operator sending the state on site i to site $i+1$, on a lattice of p_{m-1} sites with n_m (n_1) states per site for $\epsilon=-1$ $(\epsilon=+1)$ respectively.

The parameters $\lambda_i^{(k)}$ appearing at every level are solutions of the Bethe Ansatz equations:

$$\prod_{l_{k+1}=1}^{p_{k+1}}\left(\frac{x^\epsilon\sin(\lambda_{l_{k+1}}^{(k+1)}-\lambda_i^{(k)}+\gamma)}{\sin(\lambda_{l_{k+1}}^{(k+1)}-\lambda_i^{(k)})}\right)$$
$$\times\prod_{l_k=1,\,l_k\neq i}^{p_k}\left(\frac{\sin(\lambda_i^{(k)}-\lambda_{l_k}^{(k)}+\gamma)}{\sin(\lambda_i^{(k)}-\lambda_{l_k}^{(k)}-\gamma)}\right)$$
$$\times\prod_{l_{k-1}=1}^{p_{k-1}}\left(\frac{x^{-\epsilon}\sin(\lambda_i^{(k)}-\lambda_{l_{k-1}}^{(k-1)})}{\sin(\lambda_i^{(k)}-\lambda_{l_{k-1}}^{(k-1)}+\gamma)}\right)=1,$$
$$i=1,...,p_k, \quad k=1,...,m-2 \quad (37)$$

and

$$\Lambda^{(1,p_{m-1})}\prod_{l_{m-1}=1,l_{m-1}\neq i}^{p_{m-1}}\left(\frac{\sin(\lambda_i^{(m-1)}-\lambda_{l_{m-1}}^{(m-1)}+\gamma)}{\sin(\lambda_i^{(m-1)}-\lambda_{l_{m-1}}^{(m-1)}-\gamma)}\right)$$
$$\times\prod_{l_{m-2}=1}^{p_{m-2}}\left(\frac{x^{-\epsilon}\sin(\lambda_i^{(m-1)}-\lambda_{l_{m-2}}^{(m-2)})}{\sin(\lambda_i^{(m-1)}-\lambda_{l_{m-2}}^{(m-2)}+\gamma)}\right)=1,$$
$$i=1,...,p_{m-1}. \quad (38)$$

Finally, the coefficients $F^{\delta_{k_1}^{(1)},...,\delta_{k_1}^{(p_1)}}$ are such that F and is an eigenvector of the transfer matrix $\tau^{(m-1;p_1)}(\lambda;\lambda_1^{(1)},...,\lambda_{p_1}^{(1)})$. Note that, as usually happens in the ABA diagonalization, these equations imply the vanishing of the residues of $\Lambda^{(m-k;p_k)}(\lambda;\{\lambda_i^{(k)}\})$ at the $\lambda_j^{(k+1)}$'s. This was expected since the transfer matrix is non-singular at these values of the spectral parameter.

All the possible combinations of p_i's satisfying (35) should be considered. For those with a first vanishing $p_{k'}$, equations (36, 37) truncate accordingly with $\Lambda^{(m-k';p_{k'})}\equiv 1$ and all products $\prod_1^0$ set to one; equation (38) is not used. The BAE clearly display the A_{m-1} Dynkin diagram structure when one uses the shifted parameters $\nu_i^{(k)}=\lambda_i^{(k)}+k\gamma/2$.

At every level l of this diagonalization one has the option of choosing among $n_{k_0^{(l)}}$ possible pseudo-vacuum. To cover the largest number of subspaces of the Hilbert space of the chain one should consider all choices. This gives distinct eigenvectors but equal eigenvalues. This reflects a large degeneracy of the spectrum and is different from the simple models $(1,...,1;m,m)$. Another difference lies in the appearance of the eigenvalue $\Lambda^{(1,p_{m-1})}$ at the last level. As explained in [4] the diagonalization of the unit-shift operator is in principle simple. Formulae for the degeneracies of its eigenvalues have been derived by Bauer [12].

Are there eigenstates not obtained by the foregoing procedure? From the action of the C operators one infers that states lying in the subspaces $\mathcal{A}_k \otimes ... \otimes \mathcal{A}_k$, for every fixed k, are not "reached" by the Ansatz. I now fill a gap in [4] and give the action of the transfer matrix on such states. One easily derives:

$$\tau(\lambda)|\alpha_k^{(1)},...,\alpha_k^{(N)}\rangle = \left[\tau^{(n_k,N)} + \mathbb{I}\sum_{k'\neq k} n_{k'}\prod_{i=1}^{N}\left(x^{\mathrm{sign}(k'-k)}\frac{\sin(\lambda-\mu_i)}{\sin(\lambda-\mu_i+\gamma)}\right)\right] \times |\alpha_k^{(1)},...,\alpha_k^{(N)}\rangle \quad (39)$$

where $\alpha_k^{(i)} \in \mathcal{A}_k$ and $\tau^{(n_k,N)}$ is the unit-shift operator for n_k states and N sites. Note that for a given k, all the states with the same eigenvalue for $\tau^{(n_k,N)}$ have the same eigenvalue for $\tau(\lambda)$.

The rational limit of all the above equations is obtained by letting $\lambda_* \to \gamma\lambda_*$, $\mu_i \to \gamma\mu_i$, dividing the eigenvalues by $(\sin\gamma)^N$ and taking the limit $\gamma \to 0$.

For vanishing inhomogeneities, the Nth power of the eigenvalue $\Lambda^{(m,p_0)}(0,\{0\})$ is equal to one because one has the unit-shift operator for N sites and n states. One can verify this using equations (36, 37, 38).

The eigenvalues of the Hamiltonians and higher conserved quantities are easily derived by taking the logarithmic derivatives of the eigenvalue $\Lambda^{(m;p_0)}(\lambda,\{\lambda_1^{(0)},...,\lambda_N^{(0)}\})$ at vanishing spectral parameter. No closed form expressions of these derivatives are known. However, for vanishing inhomogeneities ($\mu_i = 0$), the first $N-1$ logarithmic derivatives of the eigenvalues in (39) are easily found to vanish (in both trigonometric and rational cases). The first and second derivatives, E_2 and E_3, of the logarithm of the eigenvalue $\Lambda^{(m;p_0)}$ can be easily found:

$$E_2 = \sum_{i=1}^{p_1}\frac{\sin\gamma}{\sin\lambda_i^{(1)}\sin(\lambda_i^{(1)}+\gamma)},$$
$$E_2^{rat} = \sum_{i=1}^{p_1}\frac{1}{\lambda_i^{(1)}(\lambda_i^{(1)}+1)} \quad (40)$$

and

$$E_3 = 2\sum_{i=1}^{p_1}\frac{\sin\gamma\,\cos\lambda_i^{(1)}}{\sin^2\lambda_i^{(1)}\sin(\lambda_i^{(1)}+\gamma)} - (E_2)^2,$$
$$E_3^{rat} = 2\sum_{i=1}^{p_1}\frac{1}{(\lambda_i^{(1)})^2(\lambda_i^{(1)}+1)} - (E_2^{rat})^2. \quad (41)$$

4 Conclusion

I have introduced new classes of solutions of the Yang-Baxter equation and found their symmetries. These models appear as hybrids between $su(n)$ XXZ and $su(m)$ XXZ models but share their main characterizing features with the latter models. The diagonalization of the conserved quantities was then done using the algebraic Bethe Ansatz procedure.

The issue of completeness of the Bethe Ansatz diagonalization, be it in its algebraic or coordinate form, is still an open issue for all but the simplest model, the $su(2)$ XX model. Various completeness derivations exists, however they all include some reasonable but unproven element. This should not be construed as a hindrance to the study of the thermodynamic limit in the Thermodynamic Bethe Ansatz (TBA) framework [13]. As the number of sites becomes large the Bethe Ansatz equations are transformed into a system of non-linear integral equations. To do this one assumes that the solutions of the BAE take a particular form for which their total number matches the dimensions of the Hilbert space. This form is true for most of the solutions and the results obtained are in complete agreement with other methods used to study the thermodynamic limit. This is probably due to the "fact" that the set of irregular solutions has vanishing measure.

Thus a detailed study of the spectrum in the TBA framework is desirable. Taking the logarithm of the Bethe Ansatz equations shows that the distribution of integers characterizing the solutions will get a contribution from the $\Lambda^{(1,p_{m-1})}$ correction and degeneracies associated with the integers n_k. Whether this influences the central charge and the conformal weights remains to be seen.

Another open issue is the determination of a quantum group framework. A step in this direction was taken in [5]. It should admit generalizations and would shed some light on whether multistate generalization exist for higher representations of A_m or for other Lie algebras. In particular, the multi-states A_m models should extend straightforwardly to the Lie superalgebras $su(m|n)$.

Reaction-diffusion processes in one dimension are described by a time-dependent probability distribution $P(\{\beta\},t)$ for the configuration β. This distribution obeys a stochastic master equation which can be written as a Schrödinger equation with imaginary time, $\partial_t|P(t)\rangle = -H|P(t)\rangle$ and $P(\{\beta\},t)$ are the components of the wave function in the basis of (species) states of the Hilbert space. The Hamiltonian H contains the physical transition rates. For certain processes H was found to belong to integrable hierarchies described by a Hecke algebra [14,15].

These hierarchies enter the class of models studied here. As the multi-states models are realizations of the Hecke algebra, possible physical applications may lie in the field of reaction-diffusion processes of multiple species. It would be interesting to pursue such an approach.

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